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Genera, band sum of knots and Vassiliev invariants

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Dedicated to the memory of my wife Lesia

Abstract

Recently Stoimenow showed that for every knot K and any $n \in \mathbf{N}$ and $u_0 \geqslant u(K)$ there is a prime knot K_{n,u_o} which is n-equivalent to the knot K and has unknotting number $u(K_{n,u_o})$ equal to u_0. The similar result has been obtained for the 4-ball genus g_s of a knot. Stoimenow also proved that any admissible value of the Tristram–Levine signature σ_ξ can be realized by a knot with the given Vassiliev invariants of bounded order. In this paper, we show that for every knot K with genus $g(K)$ and any $n \in \mathbf{N}$ and $m \geqslant g(K)$ there exists a prime knot L which is n-equivalent to K and has genus $g(L)$ equal to m.

MSC: 57M25

Keywords: Genus of knot; Canonical genus of knot; Vassiliev invariant; Trivalent diagram; Band sum of knots; Satellite knot; n-equivalent knots

1. Introduction

The question of what kind of geometric or topological information about knots can be detected by Vassiliev invariants (called also the invariants of finite type) is one of the most interesting in knot theory (see [4,7–9,12,14] for the discussion on this subject). In this paper, we study relationship between the classical genus of knots and Vassiliev invariants.

Let $\mathbf{N}$ denote the natural numbers. For any $n \in \mathbf{N}$, the knots K and L are called n-equivalent (rationally n-equivalent) if they cannot be distinguished by Vassiliev invariants (rational valued Vassiliev invariants) of order $\leqslant n$ [12,16]. For a given knot K, let $u(K), \sigma(K), \sigma_\xi(K)$ and $g_s(K)$ denote, respectively, unknotting number, signature, Tristram–Levine signatures and the 4-ball genus of this knot.

In [17], Stoimenow showed that for every knot K and any $n \in \mathbf{N}$ and $u_0 \geqslant u(K)$ there is a prime knot K_{n,u_o} which is n-equivalent to the knot K and has unknotting number $u(K_{n,u_o})$ equal to u_0. In [17], the similar result has been obtained for the 4-ball genus g_s of knot. Moreover Stoimenow showed that any admissible value of the Tristram–Levine signature σ_ξ (and so Murasugi signature σ) can be realized by a knot with given Vassiliev invariants of bounded order (see [17,18]). In this paper, we establish a similar result for the classical genus g of knot. More precisely, we shall show that for every knot K with genus $g(K)$ and any $n \in \mathbf{N}$ and $m \geqslant g(K)$ there exists a prime knot L which is

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n-equivalent to K and has genus $g(L)$ equal to m (Theorem 3.3). The methods and techniques we use in this paper are quite different from the ones used by Stoimenow in [17,18]. In our study of Vassiliev knot invariants, we use the classical diagrammatic approach, as in [13,14]. We also use some results and techniques of M. Eudave-Muños [5] and D. Gabai [6] concerning the band sum of knots.

In [8], Kalfagianni and Lin study n-adjacency of knots for any $n \in \mathbf{N}$. It is known that if a knot K is n-adjacent to a knot K', then K and K' are n-equivalent. The authors have proved that if a knot K is n-adjacent to a knot K' for every n, then K is isotopic to K'. This gives a partial verification of the well-known conjecture that Vassiliev invariants classify (oriented) knots. The results of Kalfagianni and Lin in [8] imply also the following. Given $n > 0$, there is a knot K with $g(K) \gg n$ that is n-adjacent to the unknot. Moreover the authors conjecture (and verify in many cases) that given a knot K of genus $g(K)$ and $n > 0$, $m > g(K)$, there is a knot K' with $g(K') > m$ such that K' is m-adjacent to K. These statements show that for a fixed $n > 0$ and K there are knots of arbitrarily high genus that are n-equivalent to K. The latter agrees with Theorem 3.3 of the present paper.

The structure of this paper is the following. In Section 2, we review briefly Vassiliev invariants, canonical and classical genera of knots and some results concerning the construction of band sum of knots. In Section 3, we consider the relationship between genus of knot and Vassiliev invariants more deeply and prove Theorem 3.3.

2. Preliminaries

Seifert provided an algorithm how to construct oriented compact surfaces S spanned by (oriented) link diagrams L. If L is a knot diagram, the algorithm yields always a connected surface. The procedure is the following [2]: smooth out all the crossings of the knot diagram, plug in discs into the resulting set of disjoint circles and connect the circles (called Seifert circles) by half-twisted bands. The resulting surface is called the canonical Seifert surface of this diagram of a knot and its genus the genus of the diagram. The canonical genus $\tilde{g}(K)$ of a knot is the minimal genus of all its diagrams. For several large classes of knots K the canonical genus $\tilde{g}(K)$ is known to be equal to the classical genus $g(K)$ (for example, for the class of homogeneous knots, which includes all alternating and positive knots [3]). However, this is not true, in general. Moreover, the difference $\tilde{g}(K) - g(K)$ can be arbitrarily large for infinitely many simple fibered knots [11].

Now consider the operation of band sum of knots. Take two unlinked knots K_1 and K_2 in S^3, join them by a thin rectangular strip ("band" or "ribbon") which meets the knots precisely in its narrow ends. The knot obtained by this process:

$$\cup \quad (knots) \quad \cup \quad \partial(band) \quad - \quad (band \quad \cap \quad knots)$$

is called a band-sum of two original knots. We use the notation $K = K_1 \#_b K_2$ for the band sum of knots K_1 and K_2 via the band b. The following inequality $g(K) \geqslant g(K_1) + g(K_2)$ obviously holds. Equality holds if and only if there exists a Seifert surface for K which is a band connected sum (using the same band) of minimal genus Seifert surfaces for K_1 and K_2 [6].

The band b in a band sum $K = K_0 \#_b K_1$ is called trivial if there is a separated sphere S for K_0 and K_1 in S^3 so that the core γ of b intersects S in a unique point. Eudave-Muños [5] has established the conditions under which the band sum of two knots is a prime knot. In particular, he proved the following:

Theorem. *(See Theorem 1 of [5].) Let $K = K_0 \#_b K_1$ be a non-trivial band sum of prime knots. If the band is disjoint from one incompressible Seifert surface for $K_0 \cup K_1$, then K is prime.*

Actually Eudave-Muños proved something more (see Remark 1 of [5]):

Let $K = K_0 \#_b K_1$ be a non-trivial band sum where K_0 and K_1 are not necessarily prime knots. Suppose b is disjoint from one incompressible surface for $K' = K_0 \cup K_1$. Then either K is prime, or there is a decomposing sphere for K disjoint from the band.

We shall use this fact in Section 3.

Let $\mathcal{K}$ be the set of oriented knots up to isotopy (knot types) in S^3 and let V be the free Abelian group freely generated by the set $\mathcal{K}$. Denote by $\mathcal{K}_n$ the subgroup of V generated by all n-singular knots, $n \geqslant 1$. The Vassiliev–Goussarov filtration of $\mathcal{K}$ is the following decreasing sequence of Abelian subgroups:

$$V \supset \mathcal{K}_1 \supset \mathcal{K}_2 \supset \mathcal{K}_3 \supset \cdots.$$

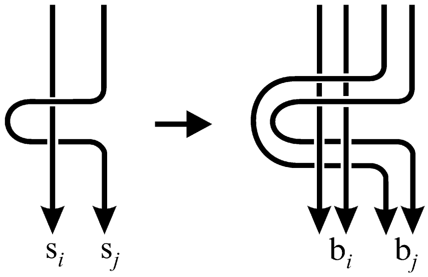

Fig. 1. The pure braid p and its double p^2.

Let H be any Abelian group and let $v:\mathcal{K}\to H$ be a map. We use the same notation v for the map $v:\mathcal{K}\to H$ and its natural extension to the homomorphism $v:V\to H$. A Vassiliev invariant of type $n\geqslant 0$ with values in an Abelian group H is a map $v:\mathcal{K}\to H$ so that its linear extension $v:V\to H$ vanishes on the subgroup $\mathcal{K}_{n+1}$. The smallest number m for which v vanishes on $\mathcal{K}_{m+1}$ is called the order of the Vassiliev invariant v. A knot K is called n-trivial if it is n-equivalent to a trivial knot O.

Denote by $\mathcal{D}_n$ the collection of trivalent diagrams of degree n and $\mathcal{A}_n$ the group of trivalent diagrams of degree n modulo the STU relations, that is $\mathcal{A}_n=\mathbf{Z}\mathcal{D}_n/\{\text{all STU relations}\}$ [13]. A trivalent diagram D is called primitive if its internal graph is connected.

Let $p\in P_{m+1}$ be any geometric pure braid with the strands $s_0,\dots,s_m$, given in the projection on the plane $\mathbf{R}^2$ and contained in the disc $B^2=I\times I$. Replace each strand s_i, $i=0,\dots,m$, by a flat band b_i that contains 2 parallel strings u_i^s, $s=1,2$, with all the under-crossings and over-crossings inherited from the braid p (see Fig. 1). The resulting pure braid $p^2\in P_{2m+2}$ will be called the double of the braid p.

We shall consider insertions of the pure braids in knots via tangle maps as in [12,16]. Any such operation on knots can be defined as a replacement of a trivial braid $1_n\in P_n$ in a fragment of a knot diagram K' with a pure braid p via a tangle map T. The orientation of the knot K determines the underlying permutation $\sigma\in S_n$ of the tangle map T [13]. Any insertion of a pure braid n-commutator $p\in LCS_n(P_{n+1})$ in a knot diagram K realizes geometrically the corresponding trivalent diagram $D\in\mathcal{A}_n$ of degree n [13]. For the background of Vassiliev knot invariants see, for example, [1].

3. Vassiliev invariants and classical genus of knot

Let K be a knot embedded in the interior of a solid torus $V\subset S^3$. We use the term "K is n-trivial inside V" if there is an embedding of K in $\mathrm{int}(V)$ and a collection of $n+1$ disjoint sets $C_1,\dots,C_{n+1}$ of crossing discs, all containing inside V, so that for each $0<m\leqslant n+1$, changing all crossings in any m of these sets simultaneously along the corresponding discs, yields an unknotted curve that can be isotoped inside a 3-ball in $\mathrm{int}(V)$.

Our aim here is to construct for each n an n-trivial prime knot K_n that has genus equal to one. We proceed as in [14]. For each $n\in\mathbf{N}$ put

$$\hat{p}_n=p_{n-1,n}\cdot p_{n-2,n-1}\cdot\dots\cdot p_{1,2}\cdot p_{0,1}\cdot p_{1,2}^{-1}\cdot\dots\cdot p_{n-2,n-1}^{-1}\cdot p_{n-1,n}^{-1}\in P_{n+1}.$$

Now let us consider a knot diagram L_n' which represents a trivial knot O in a solid torus V as indicated in Fig. 2(a). Note that the winding number $w(L_n')$ of L_n' in V is equal to zero. In the box $B\subset\mathrm{Int}(V)$ the part of the diagram L_n' is represented by the trivial braid b on $n+2$ strands. Replace first the pure braid b in B with the geometric braid $\hat{p}_n\cdot(\hat{p}_n)^{-1}\in P_{n+1}\subset P_{n+2}$, which represents a trivial braid on $n+2$ strands. This yields another diagram L_n of the knot O which is also trivial inside V. Now, for a given n, consider the one-branch trivalent diagram $T_{e_{n+1}}$ of degree $n+1$ that corresponds to the trivial permutation $e_{n+1}=(0)(1)\dots(n)$ in the symmetric group S_{n+1} (see [13]). As noted in [13], $T_{e_{n+1}}$ represents a non-trivial element of the vector space $\mathcal{A}_{n+1}\otimes\mathbf{Q}$. Applying an elementary one-branch move C_{n+1} to L_n, as indicated in Fig. 2(b), we shall obtain a knot diagram K_n' which realizes geometrically the one-branch trivalent diagram $T_{e_{n+1}}$. Note that the C_{n+1}-move we apply to L_n consists now in replacing in B the pure braid $\hat{p}_n\in P_{n+1}\subset P_{n+2}$ with the braid $\hat{p}_{n+1}\in P_{n+2}$. Since $T_{e_{n+1}}\neq 1$ in $\mathcal{A}_{n+1}\otimes\mathbf{Q}$, the diagram K_n' represents a non-trivial knot in a 3-space. By definition, K_n' is an n-trivial knot. Moreover the knot K_n' is essential in V.

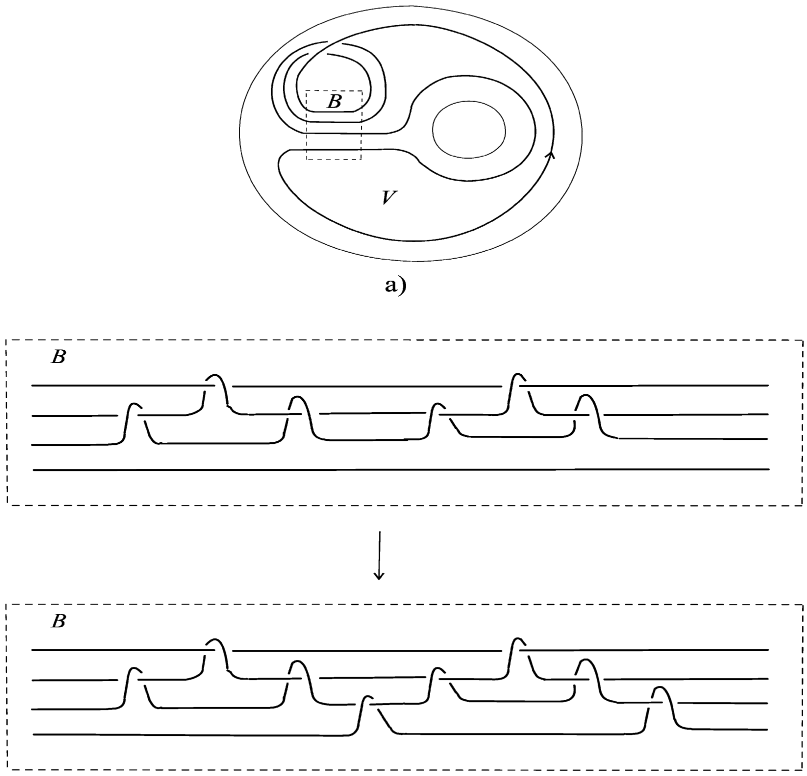

Fig. 2. An application of one-branch C_{n+1}-move to the trivial knot L_n in the solid torus V (here $n = 2$). The resulting knot is K'_n.

For each $n \in \mathbf{N}$, let K_n be the untwisted double of the knot K'_n. The knot K_n can be also considered as embedded inside the solid torus V. By Theorem 4 of [10], the knot K_n is also n-trivial inside V. Moreover, K_n has genus equal to 1 and unknotting number equal to 1, so it is prime (see, for example, [15]). We thus proved the following proposition.

Proposition 3.1. *For each $n \in \mathbf{N}$ the knot K_n constructed above is a non-trivial prime knot of genus one, which is essentially embedded in the standard solid torus V and is n-trivial inside V.*

Theorem 3.2. *For each composite knot K with genus $g(K) = g$ and any $n \in \mathbf{N}$ there is a prime knot K' which is n-equivalent to K and has genus equal to g.*

Proof. Let $K = K_1 \# K_2 \# \cdots \# K_m$ be the decomposition of the knot K into the prime summands $K_i, i = 1, \dots, m$, where $m > 1$. By induction argument, we may assume that $m = 2$ and $K = K_1 \# K_2$, where K_1 and K_2 are prime knots. Consider now a minimal Seifert surface S for K. We may also assume that $S = S_1 \#_d S_2$, where S_1 and S_2 are disjoint minimal Seifert surfaces for K_1 and K_2, respectively, and d is a trivial band in the band sum $S = S_1 \#_d S_2$. We shall represent a Seifert surface S_1, in the *disc-band form*, i.e. as a union of the disc D^2_1 and flat bands c_j glued to D^2_1, to which some full twists are added if necessary. Similarly, S_2 is represented as a union of the disc D^2_2 and flat bands d_k glued to D^2_2. Moreover the surfaces S_1 and S_2 may be chosen to be separated in $\mathbf{R}^3$ by a sphere $\tilde{S}$, i.e. S_1 is contained inside the ball bounded by $\tilde{S}$, while S_2 is contained in the exterior of this ball. Let $\Sigma_i \subset S_i$, $i = 1, 2$, be a bouquet of circles based at a point $p_i \in S_i$ that represent a simplectic base for $H_1(S_i, \mathbf{Z})$. Denote by $\beta_1, \beta_2, \dots$ the circles in the bouquet Σ_1 and $\gamma_1, \gamma_2, \dots$ the circles of the bouquet Σ_2. Each circle β_j can be considered as a core of the corresponding flat band c_j. Similarly, each circle γ_k is considered as a core of the corresponding flat band d_k. We may assume, without loss of generality, that the band d joins in S some bands c_j and d_k of the Seifert surfaces S_1 and S_2, respectively, in a trivial way.

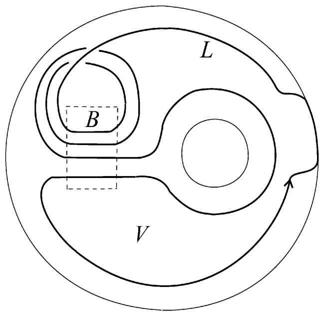

Fig. 3. The modified knot L'_n with a small arc δ contained in ∂V.

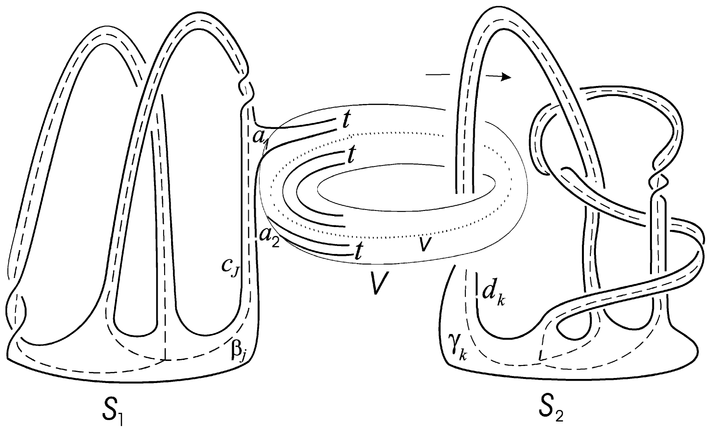

Fig. 4. A specific embedding of the solid torus V in the 3-space with respect to the Seifert surfaces S_1 and S_2.

Now our aim is to replace the band d with the other one, say u, which is also trivial in the band sum $S' = S_1 \#_u S_2$, has the same ends as d (i.e. $d \cap S_1 = u \cap S_1$ and $d \cap S_2 = u \cap S_2$), and is specified by the condition that allows an application of a doubled one-branch C_{n+1}-move to $\partial S'$. For this, consider first a copy of the solid torus V which contains the trivial knot L'_n, where both L'_n and V are defined in the proof of Proposition 3.1. We may slightly modify L'_n in V, and suggest that $\partial V \cap L'_n$ is non-empty and consists of a small arc δ, positioned outside the box B, where $B \subset V$. Let L be the arc of L'_n which remains after removing δ from L'_n (see Fig. 3). It is assumed that L intersects ∂V transversely in two points. Consider now the untwisted double $\tilde{L}$ of the arc L in V. Then $\tilde{L}$ bounds a flat band t in V to which, may be, some full twists are added. The band t intersects ∂V in two small disjoint arcs a_1 and a_2, both the sides of t. Let v be a core of V. We now place V in the 3-space so that the following conditions are satisfied:

(a) $V \cap (S_1 \cup S_2) = t \cap (\partial S_1 \cup \partial S_2) = t \cap c_j = a_1$;
(b) $\mathrm{lk}(v, \gamma_k) = \pm 1$.

By the construction, t is being glued to the surface S_1 along the side a_1, whereas the other side, a_2, is free (see Fig. 4).

To continue, let us consider another copy q of a thin band with the opposite side b_1 and b_2. We first past q to the band t by identifying the sides b_2 and a_2 of the bands and slide the free side b_1 of q along a meridian of the circle β_j to make a non-trivial link with the band c_j of the surface S_1. After that, we move the free side of q to the surface S_2 outside the solid torus V and past it to the band d_k along a small arc on ∂S_2 (see Fig. 5). As a result, the surfaces S_1 and S_2 are joined by a new "long" trivial band u, where u is obtained from the bands t and q by gluing the sides a_2 and b_2, respectively, together. We thus require that the band u intersects S_1 (S_2) only along the side a_1 (b_1, respectively).

The new band u intersects the box B in $n+2$ parallel ribbons $v_1, v_2, \ldots, v_{n+2}$, that realize the double p' of the pure braid $\hat{p}_n \cdot \hat{p}_n^{-1} \in P_{n+1} \subset P_{n+2}$. Let $\tilde{p}_n$ denote the double of the pure braid $\hat{p}_n \in P_{n+1} \subset P_{n+2}$ and let $\tilde{p}_{n+1}$ denote the

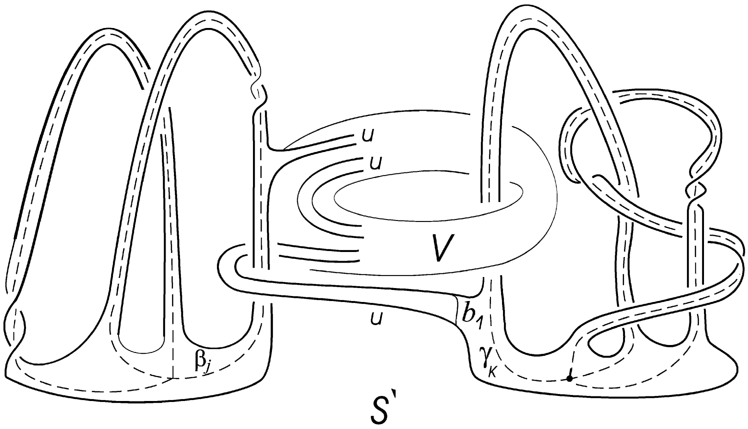

Fig. 5. The band sum S' of the Seifert surfaces S_1 and S_2 via the band u.

double of the pure braid $\hat{p}_{n+1} \in P_{n+2}$. To continue, replace the geometric braid $\tilde{p}_n \in P_{2n+4}$ in the box B with a new geometric braid $\tilde{p}_{n+1} \in P_{2n+4}$. This replacement yields a new band w joining Seifert surfaces S_1 and S_2. As can be easily seen, this replacement is actually a doubled C_{n+1}-move on the knot K, so the resulting knot K' is n-equivalent to K. Moreover, by our construction, we have $M' = S_1 \#_w S_2$, where M' is a Seifert surface for the knot $K' = \partial M'$. Notice that the application of doubled C_{n+1}-move to K is performed only on the subband t of the band u and does not change the position of its subband q. Let l denote the corresponding subband of w that is obtained from t by the given deformation. In our notation, the band w consists of two subbands, l and q, glued together along a side a_2.

Lemma 3.1. *The knot K' is a non-trivial band sum of prime knots K_1 and K_2.*

Proof. We have to show that the band w in the band sum $K' = K_1 \#_w K_2$ is non-trivial. Suppose opposite, that is there is a sphere $\tilde{P}$ in $\mathbf{R}^3$ which intersects the band w once and separates the knots K_1 and K_2. It is assumed that K_1 is contained inside the ball D bounded by the sphere $\tilde{P}$ in $\mathbf{R}^3$. Note that the sphere $\tilde{S}$ constructed above intersects the band w transversely in three arcs, first time along the band l, and the other two times along the band q that goes around the band c_j. We also may assume that the spheres $\tilde{P}$ and $\tilde{S}$ intersect transversely and their intersection consists of a finite number of circles. Now by applying the standard cut-and-past technique to the sphere $\tilde{S}$ and isotoping the band w and the surfaces S_1 and S_2, if needed, we shall obtain a new sphere and a new Seifert surface T which have the following properties:

(a) the surface $M' = S_1 \#_w S_2$ is deformed to a new band sum T of surfaces, $T = S_1' \#_h S_2'$, where h is the result of deformation of the band w;
(b) the resulting deformed sphere is $\tilde{P}$;
(c) the sphere $\tilde{P}$ intersects the band h once and does it along a side parallel to a_1.

This is impossible since the part of the band w lying in the embedded solid torus V is essential there and the remaining part of w has a non-trivial link with the band c_j. □

Now, it follows from Theorem 1 of [5], that the knot K' is prime. Moreover by [6], we have $g(K') = g(K_1) + g(K_2)$, so $g(K') = g(K)$. This completes the proof of the theorem. □

Theorem 3.3. *For every knot K with genus $g(K)$ and any $n \in \mathbf{N}$ and $m \geqslant g(K)$ there exists a prime knot L' which is n-equivalent to K and has genus $g(L')$ equal to m.*

Proof. If K is a trivial knot, then the assertion follows directly from Proposition 3.1 and Theorem 3.2. If $m = g(K)$, the assertion follows directly from Theorem 3.2. So assume that K is a non-trivial knot and let the numbers $n \in \mathbf{N}$ and $m > g(K)$ are given. By induction arguments, it suffice to construct a prime knot L' which is n-equivalent to K and has genus $g(L')$ equal to $g(K)+1$. Let $K = K_1'\#K_2'\#\cdots\#K_l'$ be a decomposition of K into prime summands K_i. We may proceed as in the proof of Theorem 3.2. First represent the minimal Seifert surface S for K as the trivial band sum of the disjoint surfaces $S_1,\dots,S_l$, where S_i is a minimal Seifert surface for the knot K_i'. We may also assume that each S_i is represented in a disc-band form (see above). Let Σ_i be the bouquet of circles for S_i, as in the proof of Theorem 3.2. Fix in each surface S_i the band c_i with the core β_i, where β_i is being considered as a circle of the bouquet Σ_i. The surfaces S_i can be considered positioned in disjoint balls $B_i, i=1,\dots,l$. Let $\tilde{S}$ be a Seifert surface of genus 1 for the knot K_n, constructed in the proof of Proposition 3.1. As before, we regard the knot K_n and the Seifert surface $\tilde{S}$ as positioned inside the standard solid torus V (see above). Let α be the core of the solid torus V.

We shall construct a knot L' via the band sum of knots K and K_n as follows. First past the band $b=b(I\times I)$ along the side $b(I\times 0)$ to the Seifert surface $\tilde{S}$. To continue, we move the free side $b(I\times 1)$ of the band b to the ball B_1 and extend it near the band c_1 along the meridian m_1 of the circle β_1 inside B_1 so that it first creates a non-trivial link with c_1 and then goes outside this ball. We may require that b intersects ∂B_1 twice, once when going inside the ball B_1, and the second time, when coming out it. Thus the set $\partial B_1 \cap b$ consists of the two arcs that are parallel to the side $b(I\times 0)$ of b. Next, we move the free side of b to the solid torus V and slide it along a meridian to make a non-trivial link with the core α.

To continue, we extend the band b in the same way as on the first step, this time however with respect to the Seifert surface S_2. More precisely, first create a non-trivial link of b with the band c_2 of the surface S_2, then come back to the solid torus V to create a non-trivial link of the band b with α. We proceed constructing the band b in the same way as before. On the last step l, we move the free side of the band b to the band c_l, then extend it along a meridian of β_l inside the ball B_l, so that first it creates a non-trivial link with c_l and then it leaves B_l. After that, we move the free end of b again to the solid torus V to create a non-trivial link with its core α and glue then the band along $b(I\times 1)$ to the surface S_l. We may require that b intersects each sphere ∂B_i, $i \leqslant l-1$, twice, while $b\cap\partial B_l$ consists of three arcs parallel to the side $b(I\times 0)$ of b. As a result, we have obtained a "long" band (denoted also by b) joining the surfaces S and $\tilde{S}$. The resulting knot L' (which bounds the resulting surface S') is a band sum of the knots K and K_n in a non-trivial way: $L' = K\,\#_b\,K_n$. Actually, the Seifert surface S' for L is obtained via the band sum of the Seifert surfaces S and $\tilde{S}$.

It is not difficult to verify that $K\,\#_b\,K_n$ is a non-trivial band sum of knots K and K_n. Moreover, for any $i\leqslant l$ there is no sphere in $\mathbf{R}^3$ that separates the knot K_i' from the remaining part of the knot $K\,\#_b\,K_n$. This follows from the definition of the knot $K\,\#_b\,K_n$ and the arguments similar to those in the proof of Lemma 3.1. Now it follows from Remark 1 of [5], that the resulting knot L' is prime. Since the knot K_n is n-trivial inside V, the knot L' is n-equivalent to K. Since the Seifert surface S' is obtained via the band sum of the minimal Seifert surfaces $\tilde{S}$ and S for the knots K_n and K, respectively, we may apply the result of Gabai cited before [6]. It follows that $g(L') = g(K)+g(K_n) = g(K)+1$, completing the proof. □

Remark 3.1. There is also another proof of Theorem 3.3. Indeed, by Theorem 3.2, we may replace the knot K with a knot K' which is prime, n-equivalent to K and has the same genus as the knot K has. After that we consider the Seifert surfaces S' and S_n of minimal genus for the knots K' and K_n, respectively, and position them in disjoint balls. Finally we take, as before, a non-trivial band sum of the surfaces S' and S_n, $\tilde{S} = S'\,\#_b\,S_n$. The resulting knot, which bounds the surface $\tilde{S}$, will be the desired prime knot L'.

We do not know if the assertion of Theorem 3.3 still holds when the term "prime" in the conclusion is replaced with that of "simple" or "band-prime". We do not also know if the same assertion holds for the canonical genus of knots, that is when "genus" is replaced with "canonical genus".

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